Published by Institute of Physics Publishing for SISSA/ISAS

Received: *October 20, 2003*
Revised: *December 3, 2003*
Accepted: *December 12, 2003*

Closed geodesics on Gödel-type backgrounds

Daniel Brace

Department of Physics, Technion, Israel Institute of Technology
Haifa 32000, Israel
E-mail: brace@physics.technion.ac.il

Abstract: We consider radial oscillations of supertube probes in the Gödel-type background which is U-dual to the compactified pp-wave obtained from the Penrose limit of the NS five-brane near horizon geometry. The supertube probe computation can be carried over directly to a string probe calculation on the U-dual background. The classical equations of motion are solved explicitly. In general, the probe is not restricted to travel unidirectionally through any global time coordinate. In particular, we find geodesics that close.

Keywords: D-branes, Superstring Vacua.

Contents

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1. Introduction

It was shown by Gauntlett et al. [1] that low energy string theory admits supersymmetric solutions of the Gödel type [2]. These homogeneous spaces have Closed Timelike Curves (CTCs) and Closed Null Curves (CNCs) which are homotopic to a point. It was then pointed out that these Closed Causal Curves (CCCs) are not present in dimensional upliftings of certain T-dual versions of these Gödel spaces [3]. The reason was understood in [4]: the uplifted spacetime is a standard pp-wave of the type that has been recently studied as the Penrose limit of certain near horizon brane geometries [5]. Such realization raised the hope that string theory could shed light on one of the more important open problems in General Relativity, or more generally, in gravitational theories: are geometries with CCCs intrinsically inconsistent or is propagation of matter in geometries with CCCs intrinsically inconsistent?

In this paper, we obtain results that suggest that string theory on certain supergravity backgrounds possessing CCCs is inconsistent. By studying the geodesics of $D2$ brane (supertube) probes on a certain Gödel-type background we are able to find classical trajectories that close on themselves in spacetime. We will show that after a finite evolution in an affine parameter, an initial slice of D2 brane probe, which may be spacelike and located within a causal region of the Gödel space, may return to its original position and that further evolution of its coordinate embedding is periodic in this affine parameter. We will refer to this type of solution of the probe equations of motion as a periodic geodesic.

A closed geodesic is then a periodic geodesic whose affine parameter has been periodically identified so that the probe's worldvolume is finite. In the case of D-brane geodesics, there are extra worldvolume gauge fields that must share a common period with the spacetime coordinates in order to make this identification. We will show that this is possible if a certain combination of parameters is rational. In particular, this combination depends on the parameter f of the Gödel background and on the radius of a compact direction, along with the magnetic field and the momentum conjugate to the electric field on the brane. It is always possible to choose the momentum conjugate to the electric field appropriately in order to obtain closed geodesics. On the other hand, if this choice is not made or if the affine parameter is simply not periodically identified, the D2-brane probe traveling along a periodic geodesic leads to divergences in the energy momentum tensor. It is clear that this renders the probe calculation inadequate, and the effects of gravitational backreation and self-interaction should be considered. At the quantum level, the closed geodesics are potentially problematic. In analogy with point particle examples where closed one dimensional geodesics exist, one might expect pathologies to manifest themselves in the full quantum theory. We leave these issues for future work. It is interesting to note that the well known particle geodesics in the Gödel Universe do not lead to the same behavior that we encounter when considering stringy matter on the same background. Recent discussions of CCCs in string theory can be found in [4] and [6]–[12].

The outline of the paper is as follows. In the first two sections 2 and 3, we solve the classical equations of motion of the supertube probe, whose radius we allow to oscillate. In section 4, by considering some specific parameters we are able to find probe geodesics that are periodic and discuss under which conditions they can be closed. We briefly discuss the gravitational coupling of the probes in section 5. The details of the solution are worked out in appendix A. In appendix B, we translate the supertube solutions into the U-dual string geodesics and make some brief comments about string quantization on the compactified pp-wave background.

2. Supertube probes

The metric and background fields of the Gödel Universe we will be studying are given by[1]

$$\begin{aligned} ds^2 &= -\left[dt + fr^2 d\theta\right]^2 + dy^2 + dr^2 + r^2 d\theta^2 + \delta_{ij} dx^i dx^j \,, \\ B_{NS} &= fr^2 dy \wedge d\theta \,, \qquad C^{(3)} = fr^2 d\theta \wedge dt \wedge dy \,, \qquad C^{(1)} = -fr^2 d\theta \,, \end{aligned} \tag{2.1}$$

which is a type-IIA supergravity background preserving one quarter of the maximal number of supersymmetries [6]. CCCs of the Gödel background can be seen by considering the curve generated by $\partial/\partial\theta$. This curve is spacelike for $r < f^{-1}$, null for $r = f^{-1}$, and timelike for $r > f^{-1}$. The surface defined by $r = f^{-1}$ will be referred to as the velocity of light surface (VLS). We will take the y direction to be compact with period $2\pi L$.

[1] In this paper, we will take f to be positive.

In flat space, it was shown [13] that cylindrical $D2$ branes can be supported against collapse by angular momentum generated by electric and magnetic fields on their world-volume, and that these supertubes are $\frac{1}{4}$ BPS. More recently, it was shown [9] that in the Gödel background (2.1) supertubes continue to exist and preserve the same supersymmetry as the background itself.

If we consider a cylindrical $D2$-brane extended in y and wrapping the θ direction N times, the system is described by a U(N) gauge theory with twisted boundary conditions. Restricting attention to the U(1) zero mode components of the field strength and radial mode, the probe is described by a Born-Infeld lagrangian with couplings to the background RR fields.

$$\mathcal{L} = -|N|\sqrt{-\det(G+\mathcal{F})} - N\, e^{\mathcal{F}} \sum C^{(n)}\,, \tag{2.2}$$

where $\mathcal{F} = F - B_{NS}$. Using static gauge, we find that the lagrangian for the supertube probe, centered at $r=0$ and extended in the θ and y directions of the background (2.1), is given by[2]

$$\mathcal{L} = -|N|\sqrt{(-\dot{r}^2 + \Delta^{-1})(r^2\Delta + \bar{B}^2) - \bar{E}^2 r^2 \Delta} - Nfr^2 + Nfr^2 E\,, \tag{2.3}$$

where

$$\Delta = 1 - f^2 r^2\,, \qquad \bar{B} = B - fr^2\,, \qquad \bar{E} = E - f\bar{B}\Delta^{-1}\,, \tag{2.4}$$

and $E = F_{0y}$ and B are the electric and magnetic fields on the brane. We choose to consider only configurations[3] with $F_{0\,\theta} = 0$. The hamiltonian, $\mathcal{H} = (N\Pi)E + (|N|P_r)\,\dot{r} - \mathcal{L}$, can be found

$$\mathcal{H} = \frac{s\,|N|}{r\Delta}\sqrt{{P_r}^2 r^2\Delta + (r^2\Delta + \bar{\Pi}^2)(r^2\Delta + \bar{B}^2)} + \frac{Nf}{\Delta}(r^2\Delta + \bar{\Pi}\bar{B})\,, \tag{2.5}$$

where

$$P_r = |N|^{-1}\frac{\partial \mathcal{L}}{\partial \dot{r}}\,, \qquad \Pi = N^{-1}\frac{\partial \mathcal{L}}{\partial E}\,, \qquad \bar{\Pi} = \Pi - fr^2\,, \qquad s = \mathrm{sign}(r^2\Delta + \bar{B}^2)\,. \tag{2.6}$$

A $D2$-brane system with nonzero field strength can be thought of as a bound state of $D2$-branes, $D0$-branes, and fundamental strings. The conjugate momentum $N\Pi$ is just the number of strings that wrap y, and NB is the number of of $D0$-branes per unit length in the y direction. For configurations where NB and $N\Pi$ are both positive, the system has a stationary solution which obeys the BPS conditions

$$r_{\mathrm{BPS}} = \sqrt{\Pi B}\,, \qquad \mathcal{H}_{\mathrm{BPS}} = N\Pi + NB\,. \tag{2.7}$$

As discussed in [12], the BPS condition, which usually gives a lower bound on $\mathcal{H}$, in some cases gives an upper bound in the Gödel background. When $s=-1$, the kinetic term for

[2] Here, as in the rest of the paper, we suppress various dimensionful parameters. We set $2\pi\alpha' = 1$, as well as $(2\pi)^{3/2}L = 1$. The latter can be recovered by taking $N \to (2\pi)^{3/2}LN$.

[3] Choosing temporal gauge (on the probe) $A_0 = 0$, one must enforce the Gauss law constraint $\partial_\theta \Pi_\theta + \partial_y \Pi_y = 0$. It is consistent to set $F_{0\,\theta} = \Pi_\theta = 0$ for vanishing $\partial_y \Pi_y$. On the other hand, as opposed to the flat space case, it is not consistent to set $E = F_{0y} = 0$ since E and Π_y are not directly proportional and differ by r dependent terms.

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the radial mode changes sign. The BPS solutions that exist with $s = -1$ actually sit at a maximum of the effective potential[4] and are stable since the kinetic term is negative. One can ask what it means for the stationary probe to be stable when it exists in the region in which $\Delta < 0$, where there are two timelike killing vectors on the worldvolume. There is perhaps a more natural way to define the stability as follows. The induced metric on the stationary probe can be diagonalized and new worldvolume coordinates introduced such that the worldvolume is foliated by spacelike Cauchy surfaces that can be evolved through a single timelike coordinate. The new coordinates will just be some linear combinations of t and θ. We can imagine this coordinate system, placed upon the cylindrical worldvolume spanned by t and θ, spiraling up the cylinder. The price to pay for this change of coordinates is the identification that follows from the periodicity of θ. Any spacelike Cauchy surface will have an infinite volume, but we can still ask whether or not the radial mode, for example, is stable as the system is evolved in the new timelike coordinate. For these purposes the coordinate identification can be ignored. We will not carry out this calculation as it is seems very unlikely, given the BPS nature of the probe, that the system is unstable in this sense either.

Notice that (2.5) is finite at the VLS if $|N\bar{\Pi}\bar{B}| = -N\bar{\Pi}\bar{B}$ and divergent otherwise. As we will see later, the quantity $-N\bar{\Pi}\bar{B}$ is proportional to the angular momentum of the probe. Probes with negative angular momentum (at the VLS) are unable to move through the VLS, while those with positive angular momentum can freely pass, assuming this is allowed by kinematics.

It was shown in [12] that the supertube probe computation is identical to a Nambu-Goto string probe computation on the background U-dual to (2.1), given by

$$\begin{aligned} ds^2 &= -dt^2 + dy^2 + 2fr^2 d\theta(dy - dt) + dr^2 + r^2 d\theta^2 + \delta_{ij} dx^i dx^j \,, \\ B &= -fr^2 d\theta \wedge (dy - dt) \,, \end{aligned} \tag{2.8}$$

where this y is compact with period $2\pi R$. This is the compactified pp-wave, which is obtained from the Penrose limit of the NS five-brane near horizon geometry. The identification of U-dual variables is as follows.

$$N \to -\omega' \,, \qquad NB \to R\omega \,, \qquad N\Pi \to p_y \,, \tag{2.9}$$

where p_y is the momentum in the y direction, w is the winding around the y direction, and w' is the non-topological winding around the θ direction. This is discussed further in appendix B.

3. Time traveling supertube probes

The hamiltonian (2.5) adequately describes the classical motion of the probe in the interior of the VLS. Yet, in many cases there appears to a problem defining the hamiltonian evolution outside the VLS. Typically, one finds that $\dot{r}$ is driven to infinity, at which point it

[4] The effective potential is defined as $\mathcal{H}$ in (2.5) with P_r set to zero

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is impossible to continue the evolution. It turns out that in these cases the choice of static gauge was inappropriate. Instead, taking (λ, ξ_1, ξ_2) to be the worldvolume coordinates on the probe, we make the the following ansatz,

$$\begin{aligned} t &= t(\lambda) \\ r &= r(\lambda) \\ y &= \xi_1 \\ \theta &= \xi_2 \\ A_{\xi_1} &= -B\xi_2 + A(\lambda) \end{aligned} \tag{3.1}$$

with all the other spacetime coordinates taken to be constants and gauge fields set to zero. Then, we find

$$\mathcal{L} = -|N|\sqrt{(-\dot{r}^2 + \dot{t}^2\Delta^{-1})(r^2\Delta + \bar{B}^2) - \bar{E}^2 r^2\Delta} - \dot{t}Nfr^2 + Nfr^2E\,, \tag{3.2}$$

where

$$\Delta = 1 - f^2r^2\,, \qquad \bar{B} = B - fr^2\,, \qquad \bar{E} = E - \dot{t}f\bar{B}\Delta^{-1}\,, \tag{3.3}$$

and now the dot indicates differentiation with respect to λ, and $E = F_{\lambda\xi_1}$ and $B = F_{\xi_1\xi_2}$ are the nonzero electromagnetic fields on the brane. Our new lagrangian can be derived using (2.2), or alternatively by inserting $\dot{t}$ s into (2.3) wherever necessary to insure a reparametrization invariant action whose lagrangian matches with (2.3) when setting $t(\lambda) = \lambda$. The coordinate λ can be thought of as a worldvolume time coordinate, but in fact as we will see shortly, the curve generated by $d/d\lambda$ on the worldvolume need not always be timelike.

$$\frac{d}{d\lambda} = \dot{t}\partial_t + \dot{r}\partial_r\,. \tag{3.4}$$

However, even when this vector is spacelike this the induced metric can remain lorentzian.

$$\sqrt{-G} = \sqrt{(-\dot{r}^2\Delta + \dot{t}^2)r^2}\,. \tag{3.5}$$

For example, when $\Delta < 0$, $\dot{t}$ can vanish and the square root of the lagrangian can remain real. One might worry whether or not a well defined Cauchy problem can be posed for small worldvolume fluctuations. Indeed this is possible, but clearly the two dimensional slices of the worldvolume determined by setting λ to a constant will not in general give spacelike Cauchy surfaces. This is related to the fact that $d/d\lambda$ need not always be timelike and to the fact that there may be more than one timelike killing vector on the worldvolume. The choice of an appropriate affine parameter that would define a timelike evolution of a spacelike surface can be made after the worldvolume embedding into spacetime, and therefore the induced metric, is known. In order to do this, the first step is to find a classical solution to the equations of motion of the probe. Solutions can be found in the usual way, by varying the fields in a lagrangian and considering the resulting equations of motion, without enforcing any particular boundary conditions and therefore without addressing a Cauchy problem, and this is what we will do in the remainder of the paper.

One can check that the ansatz (3.1) is consistent with the full equations of motion, and so we now focus on the lagrangian (3.2) and simply look for solutions to the equations of motion there. At this level, the fact that $d/d\lambda$ may not be timelike or that there may be more than one timelike killing vector on the worldvolume is irrelevant: We can consistently consider the classical evolution of the probe in terms of λ.

It will be useful to work with the routhian, $\mathcal{R} = \mathcal{L} - (N\Pi)E$, which is given by

$$\mathcal{R} = \frac{-s'|N|}{r^2\Delta}\sqrt{r^2\Delta(-\dot{r}^2+\dot{t}^2\Delta^{-1})(r^2\Delta+\bar{\Pi}^2)(r^2\Delta+\bar{B}^2)} - \dot{t}\frac{Nf}{\Delta}(r^2\Delta+\bar{\Pi}\bar{B})\,, \tag{3.6}$$

and which serves as a lagrangian for r and t. The conjugate momentum to E is defined in the same way as the previous section, and

$$s' = \mathrm{sign}(r^2\Delta+\bar{\Pi}^2)\,. \tag{3.7}$$

Since there is no explicit t dependence, the routhian defines a conserved quantity $H = -\partial\mathcal{R}/\partial\dot{t}$, which we will call the energy, given by

$$H = \frac{\dot{t}\,|N(r^2\Delta+\bar{\Pi}^2)|(r^2\Delta+\bar{B}^2)}{\Delta\big(r^2\Delta(-\dot{r}^2+\dot{t}^2\Delta^{-1})(r^2\Delta+\bar{\Pi}^2)(r^2\Delta+\bar{B}^2)\big)^{\frac{1}{2}}} + \frac{Nf}{\Delta}(r^2\Delta+\bar{\Pi}\bar{B})\,, \tag{3.8}$$

which is equal to $\mathcal{H}$ in (2.5) when $\dot{t}$ is set to one. Notice that the system is invariant under

$$\dot{t}\to-\dot{t}\,,\qquad N\to-N\,,\qquad H\to-H\,. \tag{3.9}$$

Physically, this means that a probe traveling forward (backward) in time with energy H can be interpreted as the charge conjugate[5] probe ($N\to-N$) traveling backward (forward) in time with energy $-H$. One should also be aware of the odd nature of the square root in (3.8). Although the quantity inside the square root in manifestly positive within the VLS, in the regions where $(r^2\Delta+\bar{\Pi}^2)>0$ and $(r^2\Delta+\bar{B}^2)<0$, or vice versa, the quantity is manifestly negative. Therefore, the probe is classically forbidden to enter or exist in such regions.

The equations of motion can be solved explicitly using (3.8). The details of the solution can be found in appendix A. It turns out to be more convenient to solve the the equations of motion in terms of $x=r^2$. We will now briefly review the form of the probe trajectories. When

$$H = -f^{-1}N + N(\Pi+B) \equiv H_\infty\,, \tag{3.10}$$

which is the value of H in (3.8) when $x=\infty$, the solution is given by

$$\begin{aligned} x(\lambda) &= x_t\cosh^2(\lambda)\,,\\ t(\lambda) &= \beta\sinh(\lambda)\,, \end{aligned} \tag{3.11}$$

where the coefficients x_t and β can be determined. This describes the probe in the far past contracting until it reaches the turning point x_t at which point it begins to expand and does so for the rest of its future. This turns out to be the only case when the solution is

[5] Flipping the sign of N changes the sign of the $D2$, $D0$, and $F1$ charges of the probe.

unbounded in x. All other solutions oscillate between two radial turning points.

$$
\begin{aligned}
x(\lambda) &= x_0 - \hat{x}\sin\lambda\,,\\
t(\lambda) &= T_0\lambda - \hat{T}\cos\lambda\,.
\end{aligned}
\tag{3.12}
$$

The first equation describes radial oscillation about the midpoint x_0. For $|T_0| > |\hat{T}|$, the second equation describes the probe traveling unidirectionally through time. When $|T_0| < |\hat{T}|$ the probe travels both backwards and forwards in time along different parts of its trajectory. The net drift[6] through time is determined by the magnitude and sign of T_0. In the case when T_0 vanishes, (3.12) describes a periodic geodesic.

4. Closed geodesics

The coefficients of the solution (3.12) are a bit complicated, but simplify in certain cases. When $\Pi = B$, they are given by

$$
\begin{aligned}
x_0 &= \frac{(H-H_+)(H-H_-)}{2f^2(H-H_\infty)^2}\,,\\
\hat{x} &= \frac{\sqrt{H^2(H-H_{\rm BPS})(H-H_{\text{non-BPS}})}}{2f^2(H-H_\infty)^2}\,,\\
T_0 &= \frac{sH(H-2H_\infty)}{4f(H-H_\infty)|H-H_\infty|}\,,\\
\hat{T} &= \frac{s\sqrt{H^2(H-H_{\rm BPS})(H-H_{\text{non-BPS}})}}{4f(H-H_\infty)|H-H_\infty|}\,,
\end{aligned}
\tag{4.1}
$$

where

$$
\begin{aligned}
H_{\rm BPS} &= 2NB\,,\\
H_{\text{non-BPS}} &= -2NB(1-2fB)\,,\\
H_\infty &= -f^{-1}N(1-2fB)\,,\\
H_\pm &= fNB^2 \pm \sqrt{N^2B^2[(1-fB)^2+1-2fB]}\,.
\end{aligned}
\tag{4.2}
$$

We will further restrict attention to cases when $B < 0\,,\ N < 0$, in which case $s = 1$, and we have the following hierarchy.

$$
H_{\text{non-BPS}} < H_- < 0 < H_+ < H_{\rm BPS} < H_\infty < 2H_\infty\,. \tag{4.3}
$$

When $H \leq H_{\text{non-BPS}}$, a solution exists, but $T_0 < 0$, so this should properly be interpreted as the charge conjugate probe ($N \to -N\,,\ H \to -H$) traveling forward in time, and confined within the 'potential'[7] well plotted in Figure 1b. For $H_{\text{non-BPS}} < H < H_{\rm BPS}$ some of the coefficients (4.1) become imaginary, indicating that no solutions exist in this range. At $H = H_{\rm BPS}$ we have a stationary solution. This is exactly when the probe is

[6] The solution in x and t is similar to the trajectory (in two spacelike directions) of a charged particle in a background electromagnetic field with non-zero $E\times B$. The average velocity of the charged particle is proportional to the vector $E\times B$, and the net motion is referred to as an $E\times B$ drift.

[7] Because the dynamics are described by a BI action, the 'potentials' are only useful guides when the radial momentum is small.

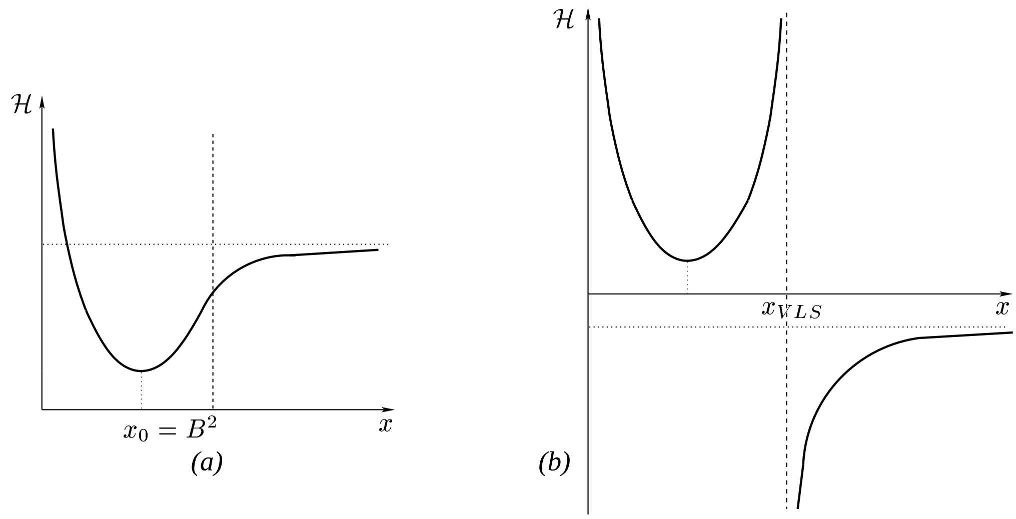

Figure 1: Effective potentials corresponding to $\Pi = B < 0$ with $N < 0$ (left) and $N > 0$ (right). The effective potential is defined as the hamiltonian $\mathcal{H}$ given in (2.5) with $P_r = 0$.

at the minimum of the 'potential' plotted in Figure 1a, with $x_0 = B^2$. As we increase H further, the solution shows oscillatory behavior in x. In the region $H_{\rm BPS} < H < H_\infty$, one can show that $T_0 > |\hat{T}|$, which implies the probe never travels backward in time. When $H = H_\infty$, the probe is no longer bound and escapes to infinity. When $H_\infty < H < 2H_\infty$, the probe has a net drift backward in time, although when it is within the VLS, it is always traveling forward in time. For $H > 2H_\infty$, the drift returns to being positive. Exactly when $H = 2H_\infty$ the orbit is periodic. In this case the coefficients simplify further.

$$\begin{aligned} x_0 &= \frac{2}{f^2} + \frac{B^2}{1-2fB}\,, \\ \hat{x} &= \frac{2(1-fB)}{f^2\sqrt{1-2fB}}\,, \\ T_0 &= 0\,, \\ \hat{T} &= \frac{(1-fB)}{f\sqrt{1-2fB}}\,. \end{aligned} \tag{4.4}$$

From these expressions we see that we can relax some of the constraints we imposed earlier. This is a valid solution whenever $B < 1/(2f)$ and $B \neq 0$. Although we have looked at a special case, by looking at the full solution in appendix A, one can see that even when $\Pi \neq B$ periodic orbits are possible. In fact, for positive H and non-vanishing Π and B whenever $N < 0$, $B < 1/(2f)$ and $\Pi < 1/(2f)$, periodic geodesics exist when

$$H = H_\infty - Nf^{-1}\sqrt{(1-2f\Pi)(1-2fB)}\,. \tag{4.5}$$

In order to determine whether or not these periodic geodesics can be closed, we must determine the period of the gauge field $A(\lambda)$ as defined in (3.1). Using a result from appendix A, we have

$$A(2\pi m) - A(0) = \int_0^{2\pi m} d\lambda E(\lambda) = -\frac{2\pi m}{2f}\left(1 + \sqrt{\frac{1-2fB}{1-2f\Pi}}\right), \tag{4.6}$$

which is a valid expression even when $\Pi \neq B$. Here, m is an arbitrary integer. Clearly this would imply that $A(\lambda)$ is not periodic, but we have yet to take into account gauge equivalence. A gauge transformation by the element $g = \exp(iyn/L)$, which is single valued[8] on the worldvolume if n is an integer, produces the following identification.

$$A \sim A + \frac{n}{L}. \tag{4.7}$$

Thus if[9]

$$\frac{L}{2f\alpha'}\left(1+\sqrt{\frac{1-2fB}{1-2f\Pi}}\right) = \frac{n}{m} \tag{4.8}$$

for some integers n and m — or in other words, if the left hand side of the above equation is rational — then $A(\lambda)$ is periodic in λ with period $2\pi m$. In this case, the affine parameter can be identified

$$\lambda \sim \lambda + 2\pi m \tag{4.9}$$

and the geodesic is closed. Since Π can be varied[10] continuously, for a fixed background and magnetic field, Π can always be adjusted so that the above quantity is rational.

5. Gravitational couplings

The probe's contribution to the energy momentum tensor can be calculated.

$$T^{\mu\nu} = \frac{-2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}. \tag{5.1}$$

It is useful to define an energy momentum density $\mathcal{T}$ on the probe through the expression

$$\sqrt{g}\, T^{\mu}_{\ \nu} = \int d^2\xi\, d\lambda\, \mathcal{T}^{\mu}_{\ \nu}\, \delta^{10}(X - X(\xi,\lambda)), \tag{5.2}$$

where X represents all the spacetime coordinates and λ and ξ parameterize the spacetime embedding of the probe. Then using the lagrangian (2.2), we find

$$\begin{aligned}
\mathcal{T}^{0}_{\ 0} &= \frac{\dot{t}^2\,|N(r^2\Delta+\bar{\Pi}^2)|(r^2\Delta+\bar{B}^2)}{\Delta\big(r^2\Delta(-\dot{r}^2+\dot{t}^2\Delta^{-1})(r^2\Delta+\bar{\Pi}^2)(r^2\Delta+\bar{B}^2)\big)^{\frac{1}{2}}} + \frac{\dot{t}Nf\,\bar{\Pi}\bar{B}}{\Delta} \\
&= \dot{t}(H - Nfr^2), \\
\mathcal{T}^{0}_{\ \theta} &= \dot{t}(-N\bar{\Pi}\bar{B}).
\end{aligned} \tag{5.3}$$

In these expressions we have already fixed the ξ parametrization, as we had done in the lagrangian (3.2).

---

[8] For $|N| > 1$ there may be gauge transformations by group elements that are not single valued, but rather are sections on a twisted bundle. These can result in 'tighter' identifications such as $A \sim A + 1/NL$.

[9] In this expression we have reintroduced an appropriate factor of $2\pi\alpha'$.

[10] At the classical level Π is not quantized, whereas B is. However, if we assume the geodesic is closed, (4.8) can be considered a quantization condition on Π, in the same way B is quantized. That is, $\int_{\mathcal{C}_\rangle} F = 2\pi n^i$, where the integral is over any worldvolume 2-cycle $\mathcal{C}_i$ and the n^i are integers.

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Let us consider a supertube traveling along a trajectory corresponding to a periodic geodesic. For the trajectories we found with $\Pi = B$ both $(H - Nfr^2) > 0$ and $(-N\bar{\Pi}\bar{B}) > 0$, indicating that ${T^0}_0$ and ${T^0}_\theta$ are positive when the probe is traveling forward in time (closer to the origin) and negative when traveling backward in time. Now consider the following integral,

$$\int_{\mathcal{N}} \sqrt{g}\, {T^0}_0 = \left(\int d^2\xi \right) \int d\lambda\, {\mathcal{T}^0}_0 \,, \tag{5.4}$$

where the first integral is over some bounded region $\mathcal{N}$ of *finite* spacetime volume containing the entire closed geodesic. The double integral over the two parameters ξ gives the constant $(2\pi)^2 L$. The integration over the last affine parameter λ will be divergent unless λ can be periodically identified. As discussed in the last section, this is possible for the case when $\Pi = B$ if $f\alpha'/L$ is rational. If this quantity is irrational or if the periodic identification is not made, then the above integral will diverge as a result of the unbounded integration over λ, and the probe approximation is not valid. The same result holds for ${T^0}_\theta$, and we will have similar results for all the other nonzero components of the energy momentum tensor.

One can also study the effects of the closed supertube geodesics in the framework suggested in [9] where a supertube domain wall sources a background which in some bounded region reproduces the Gödel background studied here. In this construction, the closed timelike curves are isolated in some shell and do not exist asymptotically. This Gödel background is U-dual to a compactified pp-wave, and the string theory on this background can be quantized [14]. In appendix B, we translate the supertube probe computation to a string probe calculation on this U-dual background. This seems like the natural place to flush out the effects of these closed geodesics at the quantum level, but we leave this for future work. We should also note that the supertubes likely decouple when the compactification radius L is taken to infinity, while in the Gödel background (2.1), the closed timelike curves continue to exist. Thus, if one is of the opinion that the closed D2 brane geodesics signal problems for string theory on this background, one still has the decompactification limit to consider. In the noncompact case, there may be suitable holographic screens [4, 11], which would ensure chronology protection.

Acknowledgments

We would like to thank Jason Fleisher and Vika Naipak for assistance. We are especially grateful to Oren Bergman, Nadav Drukker, and Shinji Hirano for useful discussions. This work was supported by the Israel Science Foundation under grant No. 101/01-1.

A. Supertube solution

Before solving the differential equation (3.8), we will first demonstrate the method of solution on a simpler example.

The harmonic oscillator. Consider the lagrangian for a harmonic oscillator

$$L = \frac{1}{2}\,\dot{x}^2 - \frac{1}{2}\,\omega^2 x^2\,. \tag{A.1}$$

The hamiltonian $H = \dot{x}p - L$, can be obtained in the usual way

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2\,, \tag{A.2}$$

and the equations of motion can be solved. However, we would like to find the harmonic oscillator solutions using a different technique.

We begin by rewriting the lagrangian in a such a way that the action is reparametrization invariant.

$$L = \frac{1}{2\dot{t}}\,\dot{x}^2 - \frac{\dot{t}}{2}\,\omega^2 x^2\,, \tag{A.3}$$

where now the dot indicates differentiation with respect to the affine parameter λ. Setting $\lambda = t$, we recover the original lagrangian. Since the lagrangian contains no explicit t dependence, there is a conserved quantity

$$H = -\frac{\partial L}{\partial \dot{t}} = \frac{1}{2\dot{t}^2}\,\dot{x}^2 + \frac{1}{2}\,\omega^2 x^2\,. \tag{A.4}$$

In order to solve for the motion, we first solve for $\dot{t}$. We find

$$\dot{t} = \frac{\pm|\dot{x}\,|}{\sqrt{2H - \omega^2 x^2}} = \frac{\pm|\dot{x}\,|}{\sqrt{p^2}}\,. \tag{A.5}$$

Notice that $p^2 \equiv 2H - \omega^2 x^2$ must be greater than or equal to zero for physical solutions. The turning points occur when $p^2 = 0$, or equivalently when $x = \pm\sqrt{2H}/\omega$. We can fix the reparametrization invariance by setting,

$$x = \frac{\sqrt{2H}}{\omega}\,\sin(\lambda)\,. \tag{A.6}$$

Plugging this back into (A.5), we find $\dot{t} = \pm\omega^{-1}$, and thus $\lambda = \pm\omega(t - t_0)$. We then arrive at the familiar result

$$x = \frac{\sqrt{2H}}{\omega}\,\sin(\pm\omega(t - t_0))\,. \tag{A.7}$$

Notice, in this example $\dot{t}$ can never vanish, so the plus and minus signs are simply related by the time reversal symmetry of the original action.

The supertube. In order to solve for the supertube motion, we can solve (3.8) for $\dot{t}$. We find

$$\dot{t} = \frac{s\bar{H}\Delta r|\dot{r}\,|}{\left(r^2\Delta\bar{H}^2 - \Delta^{-1}N^2(r^2\Delta + \bar{\Pi}^2)(r^2\Delta + \bar{B}^2)\right)^{1/2}}\,, \tag{A.8}$$

where

$$\bar{H} = H - \frac{Nf}{\Delta}(r^2\Delta + \bar{\Pi}\bar{B})\,. \tag{A.9}$$

At this point, it is convenient to use the variable $x = r^2$. Then one can show that the object inside the square root is proportional to the square of the momentum conjugate to x, in analogy with the harmonic oscillator example. Let us rewrite everything in terms of x.

$$\begin{aligned}
\dot{t} &= \frac{s\bar{H}\Delta|\dot{x}\,|}{2\sqrt{P_x^2}}\,, && \text{(A.10)}\\
P_x^2 &\equiv -(N\Pi B)^2 + x\left[(H - fN\Pi B)^2 - N^2(\Pi + B - f\Pi B)^2 + 2N^2\Pi B\right] - \\
&\quad - x^2\left[Hf + N - fN(\Pi + B)\right]^2\,, \\
\bar{H}\Delta &= (H - fN\Pi B) - xf\left[Hf + N - fN(\Pi + B)\right]. && \text{(A.11)}
\end{aligned}$$

The zeros of P_x^2 determine the radial position of the turning points where $\dot{x}$ is zero. Similarly, the zero of $\bar{H}\Delta$ determines where $\dot{t}$ is zero. Since P_x^2 must be positive, this leads to bounds on the physical values of H. Also, since P_x^2 is generically quadratic in x, it has at most two zeros. When the two zeros are identical, this corresponds to a static configuration with vanishing radial momentum. When the two zeros are distinct, the radius oscillates between them. For nonzero Π and B, P_x^2 is negative at the origin. Therefore, if P_x^2 has no zeros, P_x^2 is always negative, and this corresponds to an unphysical value of H. We can conclude that all generic solutions with nonzero Π and B perform oscillatory motion. For special values of H, the quadratic P_x^2 may reduce to a linear, in which case there is only one turning point. By considering (A.8) one can also see that P_x^2 is manifestly negative in the region where $(r^2\Delta + \bar{B}^2) > 0$ and $(r^2\Delta + \bar{\Pi}^2) < 0$, or vice versa. Therefore, both zeros occur either before or both after this physically unallowed region.

After fixing some H, one can solve for the zeros of P_x^2. Let the zeros be given by x_1 and x_2. Then defining

$$x(\lambda) = \frac{(x_1 + x_2)}{2} - \frac{(x_2 - x_1)}{2}\sin(\lambda)\,, \tag{A.12}$$

and plugging back into (A.10), $\dot{t}$ can be integrated. Of course, if there is only one turning point then we should define

$$x(\lambda) = x_1\cosh^2(\lambda)\,. \tag{A.13}$$

It is convenient to make some further definitions.

$$P_x^2 = -C^2 + bx - A^2x^2\,, \qquad \bar{H}\Delta = d - xfA\,, \tag{A.14}$$

where the new coefficients can be read off from (A.11). After setting

$$x = \frac{b}{2A^2} - \frac{\sqrt{b^2 - 4A^2C^2}}{2A^2}\sin\lambda\,, \tag{A.15}$$

we find

$$\frac{|\dot{x}\,|}{\sqrt{P_x^2}} = |A|^{-1}\,. \tag{A.16}$$

Plugging this result into (A.10), we have

$$\dot{t} = \frac{s}{4A|A|}\left((2Ad - fb) + f\sqrt{b^2 - 4A^2C^2}\ \sin\lambda\right), \tag{A.17}$$

which can be integrated to give

$$t - t_0 = \frac{s}{4A|A|}\left((2Ad - fb)\lambda - f\sqrt{b^2 - 4A^2C^2}\ \cos\lambda\right). \tag{A.18}$$

Notice in the limit $f \to 0$, we find $t = H\lambda/2|N|$, which gives the frequency of supertube radial oscillation in flat space, $\omega = 2|N|/H$. Ignoring integration constants, we write the final form of the solution,

$$\begin{aligned} x(\lambda) &= x_0 - \hat{x}\sin\lambda\,, \\ t(\lambda) &= T_0\lambda - \hat{T}\cos\lambda\,, \end{aligned} \tag{A.19}$$

where the coefficients are given by

$$\begin{aligned} x_0 &= \frac{(H - H_{4+})(H - H_{4-})}{2f^2(H - H_\infty)^2}\,, \\ \hat{x} &= \frac{\sqrt{(H - H_{\rm BPS})(H - H_1)(H - H_2)(H - H_3)}}{2f^2(H - H_\infty)^2}\,, \\ T_0 &= \frac{s(H - H_{5+})(H - H_{5-})}{4f(H - H_\infty)|H - H_\infty|}\,, \\ \hat{T} &= \frac{s\sqrt{(H - H_{\rm BPS})(H - H_1)(H - H_2)(H - H_3)}}{4f(H - H_\infty)|H - H_\infty|}\,, \end{aligned} \tag{A.20}$$

and where

$$\begin{aligned} H_{\rm BPS} &= N(\Pi + B)\,, \\ H_\infty &= -f^{-1}N + N(\Pi + B)\,, \\ H_{4\pm} &= fN\Pi B \pm \sqrt{N^2(\Pi + B - f\Pi B)^2 - 2N^2\Pi B}\,, \\ H_{5\pm} &= H_\infty \pm \sqrt{H_\infty^2 - N^2(\Pi - B)^2}\,, \\ &= H_\infty \pm |N|f^{-1}\sqrt{(1 - 2f\Pi)(1 - 2fB)}\,. \end{aligned} \tag{A.21}$$

The yet undefined quantities are given by the roots of the quartic.

$$b^2 - 4A^2C^2 = (H - H_{\rm BPS})(H - H_1)(H - H_2)(H - H_3)\,. \tag{A.22}$$

We will now derive an expression for the electric field E, which will be useful for understanding when a periodic geodesic can be closed. From the definition of the conjugate momentum Π, we have

$$E = \frac{n\,\bar{\Pi}}{r^2\Delta|r^2\Delta + \bar{\Pi}^2|}\left(r^2\Delta(-\dot{r}^2 + \dot{t}^2\Delta^{-1})(r^2\Delta + \bar{\Pi}^2)(r^2\Delta + \bar{B}^2)\right)^{\frac{1}{2}} + \frac{\dot{t}f\bar{B}}{\Delta}\,, \tag{A.23}$$

where $n = \mathrm{sign}(N)$. From this expression, we see that E diverges at the VLS if the probes angular momentum is negative, just as the hamiltonian does. Although the parametrization we chose was simple for x and t, it does not come out so nicely for E.

$$E = \frac{s\left(N\Pi B^2 + x(fB(H - H_{\rm BPS}) + N\Pi - fN\Pi B) - x^2f^2(H - H_\infty)\right)}{2fx|H - H_\infty|}\,, \tag{A.24}$$

where one must plug in the solution $x(\lambda)$. One can check that when $H = H_{\rm BPS}$ with $x = \Pi B$, the electric field satisfies $E/\dot{t} = 1$, which is the expected result for the BPS solution.

When the geodesics are periodic, T_0 vanishes. For configurations where $N < 0$, this determines that $H = H_{5+}$. Plugging this value of H into (A.24), we find

$$2\pi Y \equiv \int_0^{2\pi} d\lambda E(\lambda) = -\frac{2\pi}{2f}\left(1+\sqrt{\frac{1-2fB}{1-2f\Pi}}\right). \qquad \text{(periodic geodesics)} \tag{A.25}$$

Before obtaining this expression, we made use of the following intermediate result for $x(\lambda)$ when $H = H_{5+}$.

$$\begin{aligned}\hat{x} &= \frac{\sqrt{1-2f\Pi}(1-fB)+\sqrt{1-2fB}(1-f\Pi)}{f^2\sqrt{(1-2f\Pi)(1-2fB)}} \\ x_0 &= \frac{1}{f^2}\left(1+\frac{(1-f\Pi)(1-fB)}{\sqrt{(1-2f\Pi)(1-2fB)}}\right). \qquad \text{(periodic geodesics)}\end{aligned} \tag{A.26}$$

The gauge field A_y is periodically identified

$$A_y \sim A_y + \frac{1}{L} \tag{A.27}$$

as can be seen by considering the U(1) gauge transformation $g = \exp(\frac{iy}{L})$, which is a single valued group element. The gauge field and the spacetime coordinates are then periodic in λ if YL/α' is rational, where we reinserted an appropriate factor of $2\pi\alpha'$.

It will be useful to define a new quantity[11] p_+,

$$p_+ = -H + N(\Pi + B)\,. \tag{A.28}$$

For the case of periodic geodesics, we again set $H = H_{5+}$ and find

$$p_+ = Nf^{-1} + Nf^{-1}\sqrt{(1-2f\Pi)(1-2fB)} \qquad \text{(periodic geodesics)}\,. \tag{A.29}$$

B. U-dual string probes

As shown in [12], the D2 brane probe calculation is identical to a string probe computation on the U-dual background, which is the compactified pp-wave

$$\begin{aligned} ds^2 &= -dt^2 + dy^2 + 2fr^2 d\theta(dy - dt) + dr^2 + r^2 d\theta^2 + \delta_{ij}dx^i dx^j\,, \\ B &= -fr^2 d\theta \wedge (dy - dt)\,, \end{aligned} \tag{B.1}$$

where the y-direction is compactified with radius R. The dynamics of the string are described by the Nambu-Goto lagrangian.

$$\mathcal{L} = -\sqrt{-\det g} - B_{NS}\,, \tag{B.2}$$

[11] p_+ corresponds to the 'light-cone' momentum in the Polyakov string on the U-dual background, which we will discuss further in the next appendix B.

where g and B_{NS} are the induced metric and NS two form on the worldsheet. The U-dual of the supertube solutions are described by the following ansatz.

$$\begin{aligned} t(\lambda,\sigma) &= t(\lambda)\,, \\ y(\lambda,\sigma) &= Rw\sigma + y(\lambda)\,, \\ \theta(\lambda,\sigma) &= w'\sigma\,, \\ r^2(\lambda,\sigma) &= x(\lambda)\,, \end{aligned} \tag{B.3}$$

where σ has period 2π and λ is the world sheet time coordinate. The ansatz describes a string centered at $r = 0$ with winding w around the compact y direction and non-topological winding w' around the θ direction. The rest of the spacetime coordinates are taken to be constants. Both $t(\lambda)$ and $x(\lambda)$ can be carried over directly from the supertube solution with the following substitutions.

$$N \to -\omega'\,, \qquad NB \to R\omega\,, \qquad N\Pi \to p_y\,, \tag{B.4}$$

where p_y is the momentum conjugate to $y(\lambda)$. To get an explicit expression for $y(\lambda)$, we just need to integrate

$$y(\lambda) = y_0 + \int^{\lambda} d\lambda'\, E(\lambda')\,, \tag{B.5}$$

and make the same substitutions. Here, E is given by (A.24). When the U-dual supertube geodesics are periodic, the string geodesics generically will not be since y may not be periodic. In particular, translating the result (A.25) we find

$$Y \equiv \frac{y(2\pi) - y(0)}{2\pi} = -\frac{1}{2f}\left(1 + \sqrt{\frac{1-2f\left(\frac{-Rw}{w'}\right)}{1-2f\left(\frac{-p_y}{w'}\right)}}\right). \tag{B.6}$$

When Y/R is rational, the geodesic can be closed, otherwise the two dimensional geodesic is dense on some three dimensional surface. For example, when $p_y = Rw$ ($\Pi = B$), the string geodesics close if fR is rational.

The string theory on this background has been quantized in 'light cone' gauge [14], where the 'light cone' momentum is given by[12]

$$p_+ = -H + p_y + Rw\,. \tag{B.7}$$

When p_+ is outside of the range $-1 < fp_+ < 1$, the quantum string states belong to spectral flowed representations of the Heisenberg current algebra. The use of these representations is required to avoid the appearance of states with negative norm. For the problematic geodesics we have

$$p_+ = -w'f^{-1} - w'f^{-1}\sqrt{\left(1-2f\left(\frac{p_y}{-w'}\right)\right)\left(1-2f\left(\frac{Rw}{-w'}\right)\right)}, \tag{B.8}$$

[12] In the Polyokov framework, $p_+ = E + p_y + Rw$, but the energy E from the Polyakov formalism should be identified with $-H$ from the Nambu Goto action. Also, the f defined in [14] differs from the definition of f in this paper by a factor of 2.

where we translated the result (A.29). Since w' is a positive integer, $fp_+ < -1$, which implies the quantum states, out of which a corresponding coherent state might be constructed, should be found in spectral flowed representations.

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